

# VECTORS

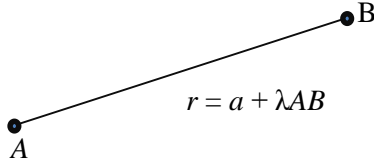
## Straight line in space

A straight line is uniquely determined in space if either; we know one point on the straight line and its direction or two points on the straight line.

### Vector equation of a line

The vector equation of a line is given by

$$r = a + \lambda AB$$



$$r = a + \lambda AB$$

$$r = a + \lambda d$$

Where;  $a$  = any point on the line

$d$  = directional vector of the line.

The Cartesian equation is given by;

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = \lambda$$

Where  $a$ ,  $b$  and  $c$  are direction vectors

### Example 1

Find the vector and Cartesian equation of a line passing through  $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$  and is parallel to  $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Solution;

$$r = a + \lambda d$$

$$r = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Cartesian equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$x = 3 + 3\lambda$$

$$y = -1 - \lambda$$

$$z = 2 + 2\lambda$$

$$\frac{x - 3}{3} = \lambda, \frac{y + 1}{-1} = \lambda, \frac{z - 2}{2} = \lambda$$

$$\frac{x - 3}{3} = \frac{y + 1}{-1} = \frac{z - 2}{2} = \lambda$$

### Example II

Find the vector and the Cartesian equation of a line passing through A(3, 4, -7) and B(1, -1, 6)

### Solution

$$r = a + \lambda d$$

$$d = AB = OB - OA$$

$$\begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 13 \end{pmatrix}$$

$$r = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -5 \\ 13 \end{pmatrix} \text{ (vector equation of line)}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -5 \\ 13 \end{pmatrix}$$

$$x = 3 - 2\lambda$$

$$y = 4 - 5\lambda$$

$$z = -7 + 13\lambda$$

$$\frac{x - 3}{-2} = \lambda$$

$$\frac{y - 4}{-5} = \lambda$$

$$\frac{z + 7}{13} = \lambda$$

Cartesian equation

$$\frac{x - 3}{-2} = \frac{y - 4}{-5} = \frac{z + 7}{13} = \lambda$$

### Example III

Find the vector and Cartesian equation of a line passing through (2, -1, 1) and is parallel to the line whose equation

$$\frac{x - 3}{2} = \frac{y + 1}{7} = \frac{z - 2}{-3} = \lambda$$

### Solution

Since the lines are parallel, it implies that they have the same parallel vectors.

$$r = a + \lambda d$$

$$r = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

Cartesian equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

$$x - 2 = 2\lambda \Rightarrow \frac{x - 2}{2} = \lambda$$

$$y + 1 = 7\lambda \Rightarrow \frac{y + 1}{7} = \lambda$$

$$z - 1 = -3\lambda \Rightarrow \frac{z-1}{-3} = \lambda$$

$$\frac{x-2}{2} = \frac{y+1}{7} = \frac{z-1}{-3} = \lambda$$

### Example III

Find the vector and Cartesian equations of the a line passing through the following points

(a) 5, -4, 6) and (3, 7, 2)

(b) (3, 4, -7) and (5, 1, 6)

#### Solution

$$r = a + \lambda AB$$

$$r = a + \lambda d$$

$$d = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 11 \\ -4 \end{pmatrix}$$

$$(a) \ r = \begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ 11 \\ -4 \end{pmatrix}$$

$$\frac{x-5}{-2} = \frac{y+4}{11} = \frac{z-6}{-4} = \lambda$$

(b) A(3, 4, -7) and B(5, 1, 6)

$$r = a + \mu d$$

$$d = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}$$

$$d = \begin{pmatrix} 2 \\ -3 \\ 13 \end{pmatrix}$$

$$r = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 13 \end{pmatrix}$$

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z+7}{13} = \lambda$$

### Example IV

Find the coordinates of the point where the line joining the points (2, 3, 1) and (3, -4 -5) meets the x-y plane

$$r = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -7 \\ -6 \end{pmatrix}$$

$$x = 2 + \lambda$$

$$y = 3 - 7\lambda$$

$$z = 1 - 6\lambda$$

For the line to meet the x-y plane,  $z = 0$

$$0 = 1 - 6\lambda$$

$$\lambda = \frac{1}{6}$$

$$x = 2 + \frac{1}{6}$$

$$x = \frac{13}{6}$$

$$y = 3 - \frac{7}{6}$$

$$y = \frac{11}{6}$$

The coordinates are  $(\frac{13}{6}, \frac{11}{6}, 0)$

### Example V

Show that  $4i - j - 12k$  lies on the line

$$r = 2i + 3j + 4k + \lambda(i - 2j + 4k)$$

#### Solution

$$r = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$(4, -1, 12)$$

$$\begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$4 = 2 + \lambda \Rightarrow \lambda = 2$$

$$-1 = 3 - 2\lambda \Rightarrow \lambda = 2 \text{ and}$$

$$12 = 4 + 4\lambda \Rightarrow \lambda = 2$$

$\therefore$  The point lies on the line since the values of  $\mu$  are the same.

### Example V

The points A, B, C have position vectors

$$\begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix}. \text{ Find which of the three points lie in the}$$

$$\text{line } r = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

#### Solution

$$r = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\text{For A, } r = \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$-4 = -1 + 3\lambda \Rightarrow \lambda = -1$$

$$5 = 4 - \lambda \Rightarrow \lambda = -1$$

$$-1 = 1 + 2\lambda \Rightarrow \lambda = -1$$

$$\Rightarrow \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} \text{ lies on the line.}$$

For B,  $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$5 = -1 + 3\lambda \Rightarrow \lambda = 2$$

$$2 = 4 - \lambda \Rightarrow \lambda = 2$$

$$3 = 1 + 2\lambda \Rightarrow \lambda = 1$$

Since the values of  $\lambda$  are not the same, point B does not lie on the line.

For C,  $\begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$8 = -1 + 3\lambda \Rightarrow \lambda = 3$$

$$1 = 4 - \lambda \Rightarrow \lambda = 3$$

$$7 = 1 + 2\lambda \Rightarrow \lambda = 3$$

$\Rightarrow$  Since the values of  $\lambda$  are the same, point C lies on the line.

### Angle between two lines

The angle between two lines is the angle between their directional vectors

**Consider two lines  $L_1$  and  $L_2$  with vector equations**

**$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}_1$  and  $\mathbf{r} = \mathbf{b} + \mu \mathbf{d}_2$  respectively**

**The angle between the two lines is given by**

**the formula  $\frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1| |\mathbf{d}_2|}$**

#### Examples

1. Find the angle between the lines;

$$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

$$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + \mu(3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

$$\cos \theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1| |\mathbf{d}_2|}$$

$$\mathbf{d}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \mathbf{d}_2 = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$$

$$\cos \theta = \frac{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}}{\sqrt{1^2 + 2^2 + 2^2} \sqrt{3^2 + 2^2 + 6^2}}$$

$$\cos \theta = \frac{3 + 4 + 12}{\sqrt{9} \sqrt{49}}$$

$$\cos \theta = \frac{19}{21}$$

$$\theta = \cos^{-1} \left( \frac{19}{21} \right)$$

$$\theta = 25.2^\circ$$

#### Example II

Find the angles between the lines

$$\frac{x+4}{3} = \frac{y+1}{5} = \frac{z+3}{4} \quad \& \quad \frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$$

**Solution**

$$\cos \theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1| |\mathbf{d}_2|}$$

$$\mathbf{d}_1 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \quad \mathbf{d}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\cos \theta = \frac{\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{3^2 + 5^2 + 4^2} \sqrt{1^2 + 1^2 + 2^2}}$$

$$\cos \theta = \frac{3 + 5 + 8}{(\sqrt{50}) \sqrt{6}}$$

$$\cos \theta = \frac{16}{\sqrt{300}}$$

$$\theta = \cos^{-1} \left( \frac{16}{\sqrt{300}} \right)$$

$$\theta = 22.5^\circ$$

#### Example III

Find the acute angle between the lines:

$$\frac{x-1}{2} = \frac{y+2}{1} = \frac{z-2}{-1} \quad \text{and} \quad \frac{1-x}{2} = \frac{y-3}{1} = \frac{z-7}{2}$$

**Solution**

$$\Rightarrow \frac{x-1}{2} = \frac{y+2}{1} = \frac{z-2}{-1} \quad \text{and} \quad \frac{x-1}{-2} = \frac{y-3}{1} = \frac{z-7}{2}$$

$$\Rightarrow \frac{x-1}{-2} = \frac{y-3}{1} = \frac{z-7}{2}$$

$$\mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{d}_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$\cos \theta = \frac{d_1 \cdot d_2}{|d_1| \cdot |d_2|}$$

$$\cos \theta = \frac{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{2^2 + 1^2 + (-1)^2} \sqrt{(-2)^2 + 1^2 + (2)^2}}$$

$$\cos \theta = \frac{-4 + 1 - 2}{\sqrt{6} \cdot \sqrt{9}}$$

⇒ The acute angle between the two lines is 47.1°

**Example IV**

Find the angle between the lines:

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-2}{1} = \lambda \quad \text{and} \quad \frac{x-5}{1} = \frac{y-1}{1} = \frac{z}{2} = \mu$$

**Solution**

$$d_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad d_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\cos \theta = \frac{d_1 \cdot d_2}{|d_1| \cdot |d_2|}$$

$$\cos \theta = \frac{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{3^2 + 2^2 + (1)^2} \sqrt{(1)^2 + 1^2 + (2)^2}}$$

$$\cos \theta = \frac{3 + 2 + 2}{\sqrt{14} \cdot \sqrt{6}}$$

$$\theta = 40.2^\circ$$

⇒ The acute angle between the two lines is 40.2°

**Note: If two lines are perpendicular, then  $(d_1 \cdot d_2) = 0$**

**Point of Intersection of two Lines**

**Example**

Find the point of intersection of the lines

$$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} \quad \& \quad \frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$$

**Solution**

$$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} = \lambda \dots\dots\dots (i)$$

$$\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1} = \mu \dots\dots\dots (ii)$$

From equation (i)

$$x = \lambda \dots\dots\dots (iii)$$

$$\frac{y+2}{2} = \lambda$$

$$y + 2 = 2\lambda$$

$$y = 2\lambda - 2 \dots\dots\dots (iv)$$

$$\frac{z-5}{-1} = \lambda$$

$$z - 5 = -\lambda$$

$$z = -\lambda + 5 \dots\dots\dots (v)$$

From equation (ii)

$$x = -\mu + 1 \dots\dots\dots (vi)$$

$$y + 3 = -3\mu$$

$$y = -3\mu - 3 \dots\dots\dots (vii)$$

$$z = \mu + 4 \dots\dots\dots (viii)$$

$$\lambda = -\mu + 1 \dots\dots\dots (*)$$

$$2\lambda - 2 = -3\mu - 3$$

$$2\lambda + 3\mu = -1 \dots\dots\dots (**)$$

Substituting Eqn (\*) in Eqn (\*\*)

$$2(1 - \mu) + 3\mu = -1$$

$$2 - 2\mu + 3\mu = -1$$

$$2 + \mu = -1$$

$$\mu = -3$$

$$\lambda = -\mu + 1$$

$$\lambda = 3 + 1$$

$$\lambda = 4$$

Equating Eqn (v) and Eqn (viii)

$$-\lambda + 5 = \mu + 4$$

$$-4 + 5 = -3 + 4$$

$$1 = 1$$

The two lines intersect

$$x = 4$$

$$y = 2\lambda - 2$$

$$y = 8 - 2$$

$$y = 6$$

$$z = -4 + 5$$

$$z = -4 + 5 = 1$$

The point of intersection of the lines is (4, 6, 1)

**Example II**

Find the point of intersection of the line

$$r = i - 2j + 3k + \lambda(2i + j - k)$$

$$r = -i + 3j + 7k + \mu(-2i + j + 2k)$$

**Solution**

From  $r = i - 2j + 3k + \lambda(2i + j - k)$

$$r = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -2 + \lambda \\ 3 - \lambda \end{pmatrix} \dots\dots\dots (1)$$

$$r = -i + 3j + 7k + \mu(-2i + j + 2k)$$

$$\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$(\mathbf{r}) = \begin{pmatrix} -1 - 2\mu \\ 3 + \mu \\ 1 + 2\mu \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 - 2\mu \\ 3 + \mu \\ 7 + 2\mu \end{pmatrix} \dots \dots \dots (2)$$

Equating the corresponding x components:

$$1 + 2\lambda = -1 - 2\mu$$

$$2\lambda + 2\mu = -2$$

$$\lambda + \mu = -1 \dots \dots \dots (3)$$

Equating the corresponding y components:

$$-2 + \lambda = 3 + \mu$$

$$\lambda - \mu = 5 \dots \dots \dots (4)$$

Equating the corresponding z component;

$$3 - \lambda = 7 + 2\mu$$

$$2\mu + \lambda = -4 \dots \dots \dots (5)$$

Eqn (3) - eqn (4)

$$2\mu = -6$$

$$\mu = -3$$

From Eqn (4)

$$\lambda - (-3) = 5$$

$$\lambda = 2$$

Substituting  $\lambda = 2$  and  $\mu = -3$  in Eqn (5);

$\Rightarrow$  The two lines intersect at (5, 0, 1)

### Example III

Find the point of intersection of the lines

$$x - 2 = \frac{y + 3}{4} = \frac{z - 5}{2} \quad \& \quad \frac{x - 1}{-1} = \frac{y - 8}{1} = \frac{z - 3}{-2}$$

**Solution**

$$x - 2 = \frac{y + 3}{4} = \frac{z - 5}{2} = \lambda \dots \dots \dots (*)$$

$$\frac{x - 1}{-1} = \frac{y - 8}{1} = \frac{z - 3}{-2} = \mu \dots \dots \dots (**)$$

From equation (\*)

$$x - 2 = \lambda$$

$$x = 2 + \lambda \dots \dots \dots (1)$$

$$y + 3 = 4\lambda$$

$$y = 4\lambda - 3 \dots \dots \dots (2)$$

$$z - 5 = 2\lambda$$

$$z = 2\lambda + 5 \dots \dots \dots (3)$$

From equation (\*\*)

$$x - 1 = -\mu$$

$$x = 1 - \mu \dots \dots \dots (4)$$

$$y - 8 = \mu$$

$$y = \mu + 8 \dots \dots \dots (5)$$

$$z - 3 = 2\mu$$

$$z = 2\mu + 3 \dots \dots \dots (6)$$

Equating the corresponding components

$$2 + \lambda = 1 - \mu$$

$$\mu + \lambda = -1 \dots \dots \dots (7)$$

$$\mu + 8 = 4\lambda - 3$$

$$\mu - 4\lambda = -11 \dots \dots \dots (8)$$

Eqn(8) - (7)

$$-5\lambda = -10$$

$$\lambda = 2$$

Substitute  $\lambda = 2$  in Eqn (8)

$$\mu - 4 \times 2 = -11$$

$$\mu = -3$$

$\therefore$  The point of intersection is (4, 5, 9)

## PLANES

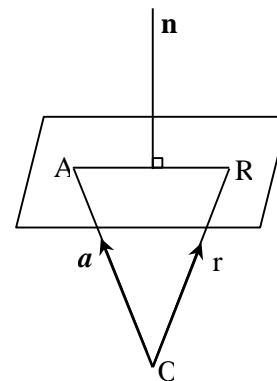
A plane is a surface which contains at least three non-collinear points. If two points are taken then the lines joining the two lines lies completely on the surface of the plane.

A plane is completely known if we know one point that lie on the plane and then the normal to the plane.

### Equation of a Plane

Suppose a plane P passes through a point A with a position vector  $\mathbf{a}$  and is perpendicular to vector  $\mathbf{n}$ . Let  $\mathbf{r}$  be any point (x, y, z) in the plane.

If two lines are perpendicular, dot product of their direction vector = 0



$$AR \cdot n = 0$$

$$(AO + OR) \cdot n = 0$$

$$(-\mathbf{a} + \mathbf{r}) \cdot \mathbf{n} = 0$$

$$(-\mathbf{n} \cdot \mathbf{a} + \mathbf{n} \cdot \mathbf{r}) = 0$$

$$\mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{r}$$

Equation of a plane is given by  $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a}$

Where  $\mathbf{n}$  = normal and  $\mathbf{a}$  = the point that lies on the plane.

### Example I

Find the equation of a plane passing through (1, 2, 3), and is perpendicular to vector  $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$

**Solution**

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a}$$

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$4x + 5y + 6z = 4 + 10 + 18$$

$$4x + 5y + 6z = 32$$

### Example II

Find the equation of a plane which contains A with position vector  $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$  and is perpendicular to  $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ .

**Solution**

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a}$$

$$\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

$$x + 2y - 2z = 3 + 8 - 4$$

$$x + 2y - 2z = 7$$

### Example III

Find the equation of a plane passing through a point A with a position vector  $-2\mathbf{i} + 4\mathbf{k}$  and is perpendicular to the vector  $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ .

**Solution**

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a}$$

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$$

$$x + 3y - 2z = -2 + 0 - 8$$

$$x + 3y - 2z = -10$$

$$x + 3y - 2z + 10 = 0$$

## Angle between two planes

The angle between two planes is the angle between their normals

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|}$$

### Example I

Find the angle between the planes  $2x + 3y + 5z = 7$ ,  $3x + 4y - z = 8$

**Solution**

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$$

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|}$$

$$\cos \theta = \frac{\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}}{\sqrt{2^2 + 3^2 + 5^2} \cdot \sqrt{3^2 + 4^2 + 1^2}}$$

$$\cos \theta = \frac{6 + 12 - 5}{\sqrt{38} \cdot \sqrt{26}} = \frac{13}{\sqrt{38} \cdot \sqrt{26}}$$

$$\theta = \cos^{-1} \frac{13}{\sqrt{38} \cdot \sqrt{26}}$$

$$\theta = 65.6^\circ$$

### Example II

Find the angle between the planes  $3x - 3y - z = 0$  and  $x + 4y - 2z = 4$

**Solution**

$$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|}$$

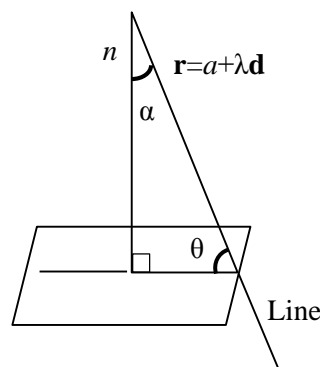
$$\cos \theta = \frac{\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}}{\sqrt{3^2 + (-3)^2 + (-1)^2} \cdot \sqrt{1^2 + 4^2 + (-2)^2}}$$

$$\cos \theta = \frac{3 - 12 + 2}{\sqrt{19} \cdot \sqrt{21}} = \frac{-7}{\sqrt{21} \cdot \sqrt{19}}$$

$$\theta = \cos^{-1} \left( \frac{-7}{\sqrt{21} \cdot \sqrt{19}} \right)$$

$$\theta = 69.5^\circ$$

## Angle between a line and a plane



$$\mathbf{n} \cdot \mathbf{d} = |\mathbf{n}| |\mathbf{d}| \cos \alpha$$

$$\theta + 90^\circ + \alpha = 180^\circ$$

$$\theta + \alpha = 90^\circ$$

$$\alpha = 90^\circ - \theta$$

$$n \cdot d = |n||d| \cos(90^\circ - \theta)$$

$$n \cdot d = |n||d| \sin \theta$$

$$\sin \theta = \frac{n \cdot d}{|n||d|}$$

$$\sin \theta = \frac{n \cdot d}{|n||d|}$$

**Example**

Find the angle between the lines

$r = i + 2j - 2k + \mu(i - j + k)$  and the plane  $2x - y + z = 4$

**Solution**

$$\sin \theta = \frac{n \cdot d}{|n||d|}$$

$$\sin \theta = \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{1^2 + (-1)^2 + 1^2} \cdot \sqrt{2^2 + (-1)^2 + 1^2}}$$

$$\sin \theta = \frac{2 + 1 + 1}{\sqrt{3} \cdot \sqrt{6}}$$

$$\sin \theta = \left( \frac{4}{\sqrt{18}} \right)$$

$$\theta = \sin^{-1} \left( \frac{4}{\sqrt{18}} \right)$$

$$\theta = 70.5^\circ$$

Find the acute angle between the line

$\frac{x-1}{-1} = \frac{y-8}{1} = \frac{z-3}{-2}$  and  $7x - y + 5z = -5$

**Solution**

$$\sin \theta = \frac{n \cdot d}{|n||d|}$$

$$\sin \theta = \frac{\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}}{\sqrt{5^2 + (-1)^2 + 1^2} \cdot \sqrt{7^2 + (-1)^2 + 5^2}}$$

$$\sin \theta = \frac{35 + 1 + 5}{\sqrt{27} \cdot \sqrt{75}}$$

$$\sin \theta = \left( \frac{41}{\sqrt{2025}} \right)$$

$$\theta = \sin^{-1} \left( \frac{41}{\sqrt{2025}} \right)$$

$$\theta = 65.7^\circ$$

**Solution**

Find the angle between the line  $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$  and  $x + y + z = 12$

**Solution**

$$\sin \theta = \frac{n \cdot d}{|n||d|}$$

$$\sin \theta = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}}{\sqrt{1^2 + 1^2 + 1^2} \cdot \sqrt{2^2 + 5^2 + 1^2}}$$

$$\sin \theta = \frac{2 + 5 - 1}{\sqrt{3} \cdot \sqrt{30}}$$

$$\sin \theta = \left( \frac{6}{\sqrt{90}} \right)$$

$$\theta = \sin^{-1} \left( \frac{6}{\sqrt{90}} \right)$$

$$\theta = 39.2^\circ$$

**Point of intersection of a line and a plane**

**Example I**

Find the point of intersection of the line  $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$  and  $x + y + z = 19$

**Solution**

$$\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1} = \lambda \dots\dots\dots (*)$$

From (\*)

$$x + 1 = 2\lambda$$

$$x = 2\lambda - 1 \dots\dots\dots (1)$$

$$y - 3 = 5\lambda$$

$$y = 3 + 5\lambda \dots\dots\dots (2)$$

$$z + 1 = -\lambda$$

$$z = -1 - \lambda \dots\dots\dots (3)$$

$$x + y + z = 12$$

$$(2\lambda - 1) + (3 + 5\lambda) + (-1 - \lambda) = 12$$

$$4\lambda = 16$$

$$\lambda = 4$$

From equation (1)

$$x = 2(4) - 1 = 7$$

From equation (2)

$$y = 5(4) + 3 = 23$$

From equation (3)

$$z = -1 - 4 = -5$$

∴ The point of intersection (7, 23, -5)

**Example II**

Find the point of intersection of the line  $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$  and

the plane  $3x + 4y + 2z = 25$

**Solution**

$$\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4} = \lambda \dots\dots\dots (*)$$

$$x = 5\lambda \dots\dots\dots (1)$$

$$\begin{aligned}
y + 2 &= 2\lambda \\
y &= 2\lambda - 2 \dots\dots\dots (2) \\
z - 1 &= 4\lambda \\
z &= 4\lambda + 1 \dots\dots\dots (3) \\
3x + 4y + 2z &= 25 \\
3(5\lambda) + 4(2\lambda - 2) + 2(4\lambda + 1) &= 25 \\
15\lambda + 8\lambda - 8 + 8\lambda + 2 &= 25 \\
31\lambda &= 25 + 6 \\
31\lambda &= 31 \\
\lambda &= 1
\end{aligned}$$

$$x = 5, \quad y = 2 - 2 = 0, \quad z = 5$$

∴ The point of intersection = (5, 0, 5)

**Example**

Find the point of intersection of the line;  $\frac{x+2}{-1} = \frac{y-2}{2} = z - 4$  and the plane  $2x - y + 3z = 10$

**Solution**

$$\begin{aligned}
\frac{x + 2}{-1} &= \frac{y - 2}{2} = z - 4 = \lambda \\
x &= -\lambda - 2 \dots\dots\dots (1) \\
y &= 2\lambda + 2 \dots\dots\dots (2) \\
z &= \lambda + 4 \dots\dots\dots (3) \\
2x - y + 3z &= 10 \\
2(-\lambda - 2) - (2\lambda + 2) + 3(\lambda + 4) &= 10 \\
-2\lambda - 4 - 2\lambda - 2 + 3\lambda + 12 &= 10 \\
-4\lambda + 3\lambda + 6 &= 10 \\
-\lambda &= 4 \\
\lambda &= -4 \\
x &= -4 - 2 = -6, \quad y = -8 + 2 = -6, \\
z &= -4 + 4 = 0
\end{aligned}$$

The point of intersection (-6, -6, 0)

**Perpendicular distance of a point from a plane**

The perpendicular distance of a point  $(x_1, y_1, z_1)$  from the plane  $ax + by + cz + d = 0$  is given by the formula;

$$D = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$$

**Example**

Find the distance of a point  $(-2, 0, 6)$  from the plane  $2x - y + 3z = 21$

**Solution**

$$D = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$x_1, y_1, z_1 = (-2, 0, 6)$$

Comparing  $ax + by + cz + d = 0$  with

$$\begin{aligned}
2x - y + 3z - 21 &= 0; \\
a &= 2, \quad b = -1, \quad c = 3, \quad d = -21 \\
D &= \frac{|-4 + 0 + 18 - 21|}{\sqrt{2^2 + (-1)^2 + 3^2}} \\
D &= \frac{-7}{\sqrt{4+1+9}} = \frac{-7}{\sqrt{14}} \text{ Units}
\end{aligned}$$

**Line of intersection of two planes**

Two planes intersect in a line

**Examples I**

Find the line of intersection of the planes  $2x + 3y + 4z = 1$  and  $x + y + 3z = 0$

**Solution**

$$\begin{aligned}
2x + 3y + 4z &= 1 \\
x + y + 3z &= 0
\end{aligned}$$

Let  $z = \lambda$

$$2x + 3y = 1 - 4\lambda \dots\dots\dots (1)$$

$$x + y = -3\lambda \dots\dots\dots (2)$$

Eqn (2) × 2

$$2x + 2y = -6\lambda \dots\dots\dots (3)$$

Eqn (1) – Eqn (3);

$$y = 1 + 2\lambda$$

$$\frac{y - 1}{2} = \lambda$$

From Eqn (2);

But  $y = 1 + 2\lambda$

$$x + y = -3\lambda$$

$$x + 1 + 2\lambda = -3\lambda$$

$$x + 1 = -3\lambda - 2\lambda$$

$$x + 1 = -5\lambda$$

$$\frac{x + 1}{-5} = \lambda$$

$$\frac{x + 1}{-5} = \frac{y - 1}{2} = z = \lambda$$

**Example II**

Find the line of intersection of planes  $2x + 3y - z = 4$  and  $x - y + 2z = 5$ .

**Solution**

$$2x + 3y - z = 4$$

$$x - y + 2z = 5$$

Let  $z = \lambda$

$$2x + 3y - \lambda = 4$$

$$x - y + 2\lambda = 5$$

$$2x + 3y = 4 + \lambda \dots\dots\dots (i)$$

$$x - y = 5 - 2\lambda \dots\dots\dots (ii)$$



Multiply Eqn (ii) by 3;  
 $3x - 3y = 15 - 6\lambda \dots\dots\dots$  (iii)

Eqn (iii) + Eqn (i);  
 $5x = 19 - 5\lambda$

$5\lambda = -x + 19$

$\lambda = -x + \frac{19}{5}$

$\lambda = \frac{(x - \frac{19}{5})}{-1}$

Multiply Eqn (ii) by 2;  
 $2x - 2y = 10 - 4\lambda \dots\dots\dots$  (iv)

Eqn (iv) - Eqn (i);  
 $-5y = 6 - 5\lambda$

$5\lambda = -6 + 5y$

$\lambda = \frac{-6}{5} + y$

$\lambda = \frac{y - \frac{6}{5}}{1}$

$x - \frac{19}{5} = \frac{y - \frac{6}{5}}{1} = z = \lambda$

**Example**

Find the Cartesian equation of a line of intersection of the lines.

$2x - 3y - z = 1$

$3x + 4y + 2z = 3$

Let  $x = \lambda$

$-3y - z = 1 - 2\lambda \dots\dots\dots$  (i)

$4y + 2z = 3 - 3\lambda \dots\dots\dots$  (ii)

Eqn (i)  $\times 2$

$-6y - 2z = 2 - 4\lambda \dots\dots\dots$  (iii)

Eqn (iii) + Eqn (ii)

$-2y = 5 - 7\lambda$

$-2y - 5 = -7\lambda$

$\frac{-2y - 5}{-7} = \lambda$

$\frac{-2(y + \frac{5}{2})}{-7} = \lambda$

Eqn (i)  $\times 4$

$\Rightarrow -12y - 4z = 4 - 8\lambda \dots\dots\dots$  (iv)

Eqn (ii)  $\times 3$

$12y + 6z = 9 - 9\lambda \dots\dots\dots$  (v)

Eqn (iv) + Eqn (v)

$2z = 13 - 17\lambda$

$\frac{2z - 13}{-17} = \lambda$

$\frac{2(z - \frac{13}{2})}{-17} = \lambda$

$x = \frac{(y + \frac{1}{2})}{\frac{7}{2}} = -\frac{(z - \frac{13}{2})}{\frac{17}{2}} = \lambda$

$x = \frac{(y + \frac{1}{2})}{\frac{7}{2}} = \frac{(z - \frac{13}{2})}{-\frac{17}{2}} = \lambda$

**Equation of a Plane**

Given three points on the plane, we can find the equation of a plane;

**Example I**

Find the Cartesian equation of a plane passing through A (0, 3, -4) B (2, -1, 2) and C (7, 4, -1)

**Solution**

Let the normal =  $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$AB = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$

$AC = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = 0$

$2p - 4q + 6r = 0$

$p - 2q + 3r = 0 \dots\dots\dots$  (i)

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} = 0$

$7p + q + 3r = 0 \dots\dots\dots$  (ii)

From (i)

$p = 2q - 3r \dots\dots\dots$  (iii)

$\Rightarrow 7(2q - 3r) + q + 3r = 0$

$14q - 21r + q + 3r = 0$

$15q - 18r = 0$

$5q - 6r = 0$

$5q = 6r$

$q = \frac{6}{5}r \dots\dots\dots$  (iv)

$\Rightarrow p = 2\left(\frac{6r}{5}\right) - 3r$

$p = \frac{12}{5}r - 3r$

$$p = -\frac{3}{5}r$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -3r/5 \\ 6r/5 \\ r \end{pmatrix} = \frac{r}{5} \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$$

$$\therefore n = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$$

$$n \cdot r = n \cdot a$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$$

$$-3x + 6y + 5z = 0 + 18 - 20$$

$$-3x + 6y + 5z = -2$$

$$3x - 6y - 5z - 2 = 0$$

### Example II

Find the equation of a plane passing through points P(4, 2, 3), Q(5, 1, 4) and R(-2, 1, 1).

**Solution**

Let the normal to the plane be  $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$PQ = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$PR = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$p - q - r = 0 \dots\dots\dots (i)$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$-6p - q - 2r = 0$$

$$6p + q + 2r = 0 \dots\dots\dots (ii)$$

From Eqn (i);

$$p = q - r$$

$$6(q - r) + q + 2r = 0$$

$$6q - 6r + q + 2r = 0$$

$$7q - 4r = 0$$

$$7q = 4r$$

$$q = \frac{4r}{7}$$

$$p = \frac{4r}{7} - r$$

$$p = \frac{-3r}{7}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -3r/7 \\ 4r/7 \\ r \end{pmatrix} = \frac{r}{7} \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix}$$

$$n = \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix}$$

$$n \cdot r = n \cdot a$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

$$-3x + 4y + 7z = -12 + 8 + 21$$

$$-3x + 4y + 7z = 17$$

$$3x - 4y - 7z + 17 = 0$$

### Example III

Find the equation of the planes passing through the following points:

- (i) A (0, 2, -4) B (2, 0, 2) C (-8, 4, 0)

**Solution**

Let the normal  $n = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$AB = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}$$

$$AC = \begin{pmatrix} -8 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix} = 0$$

$$2p - 2q + 6r = 0$$

$$p - q + 3r = 0 \dots\dots\dots (i)$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 2 \\ 4 \end{pmatrix} = 0$$

$$-8p + 2q + 4r = 0$$

$$-4p + q + 2r = 0 \dots\dots\dots (ii)$$

$$p - q + 3r = 0$$

$$p = q - 3r$$

$$-8(q - 3r) + 2q + 4r = 0$$

$$-8q + 24r + 2q + 4r = 0$$

$$-6q + 28r = 0$$

$$6q = 28r$$

$$q = \frac{14r}{3}$$

$$p = \frac{14r}{3} - 3r = \frac{5r}{3}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 5r/3 \\ 14r/3 \\ r \end{pmatrix} = \frac{r}{3} \begin{pmatrix} 5 \\ 14 \\ 3 \end{pmatrix}$$

$$n = \begin{pmatrix} 5 \\ 14 \\ 3 \end{pmatrix}$$

$$n \cdot r = n \cdot a$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 14 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix}$$

$$5x + 14y + 3z = 0 + 28 - 12$$

$$5x + 14y + 3z - 16 = 0$$

(ii) A (-1, 0, 1), B(3, 3, -2), C(-1, 1, 1)

$$\text{Let the normal} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$AB = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}$$

$$AC = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix} = 0$$

$$4p + 3q - 3r = 0 \dots\dots\dots (i)$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$q = 0$$

Substitute  $q = 0$  in Eqn (i);

$$4p = 3r$$

$$p = \frac{3r}{4}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 3r/4 \\ 0 \\ r \end{pmatrix} = \frac{r}{4} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$n = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$n \cdot r = n \cdot a$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$3x + 4z = -3 + 4$$

$$(3x + 4z = 1)$$

$$3x + 4z - 1 = 0$$

**Example IV**

Find the Cartesian equation of a plane containing the point (1, 3, 1) and it's parallel to vectors (1, -1, -3) and (2, 1, -3)

**Solution**

$$AB = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \text{ and } AC = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\text{Let the normal} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 0$$

$$p - q + 3r = 0 \dots\dots\dots (i)$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 0$$

$$2p + q - 3r = 0 \dots\dots\dots (ii)$$

$$p = q - 3r$$

$$2(q - 3r) + q - 3r = 0$$

$$2q - 6r + q - 3r = 0$$

$$3q - 9r = 0$$

$$q = 3r$$

$$p = 3r - 3r$$

$$p = 0$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 3r \\ r \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = r \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$r \cdot n = n \cdot a$$

$$n = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$3y + z = 10$$

**Example V**

Find the Cartesian equation of the plane passing through the points A(1, 0, -2), B (3, -1, 1) parallel to the line

$$r = 3i + (2\alpha - 1)j + (5 - \alpha)k$$

**Solution:**

$$r = 3i + 2\alpha j - j + 5k - \alpha k$$

$$r = 3i - j + 5k - \alpha(0j + 2j - k)$$

$$AB = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$AC = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0$$

$$2p - q + 3r = 0 \dots\dots\dots (i)$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 0$$

$$2q - r = 0 \dots\dots\dots (ii)$$

From Eqn (ii);

$$\Rightarrow r = 2q$$

$$2p - q + 3(2q) = 0$$

$$2p - q + 6q = 0$$

$$2p + 5q = 0$$

$$p = \frac{-5}{2}q$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{-5q}{2} \\ q \\ 2q \end{pmatrix} = \frac{q}{2} \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$$

$$n = \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$$

$n \cdot r = n \cdot a$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$-5x + 2y + 4z = -5 - 8$$

$$(-5x + 2y + 4z = -13)$$

$$5x - 2y - 4z - 13 = 0$$

### Example VI

Find the equation of the plane containing line

$$r = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \text{ and is parallel to the line}$$

$$r = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad AC = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \quad n = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0 ;$$

$$-2p + q - r = 0$$

$$\Rightarrow 2p - q + r = 0 \dots\dots\dots (i)$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$-p + q + 2r = 0 \dots\dots\dots (ii)$$

From Eqn (i);

$$r = -2p + q$$

$$\Rightarrow p - q - 2(q - 2p) = 0$$

$$p - q - 2q + 4p = 0$$

$$5p - 3q = 0$$

$$p = \frac{3q}{5}$$

$$r = -2\left(\frac{3q}{5}\right) + q$$

$$r = \frac{-q}{5}$$

$$n = \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{3q}{5} \\ q \\ \frac{-q}{5} \end{pmatrix} = \frac{q}{5} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

$n \cdot r = n \cdot a$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$3x + 5y - z = 3 - 5 + 0$$

$$3x + 5y - z = -2$$

### Example VII

Find the Cartesian equation of the plane formed by the lines  $r = -2i + 5j - 11k + \lambda(3i + j + 3k)$  and

$$r = 8i + 9j + \lambda(4i + 2j + 5k)$$

**Solution**

$$\text{Let } n = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Rightarrow \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$3p + q + 3r = 0 \dots\dots\dots (i)$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = 0$$

$$4p + 2q + 5r = 0 \dots\dots\dots (ii)$$

From Eqn (i);

$$q = -3p - 3r$$

$$4p + 2(-3p - 3r) + 5r = 0$$

$$4p - 6p - 6r + 5r = 0$$

$$-2p - r = 0$$

$$r = -2p$$

$$q = -3p - 3(-2p)$$

$$q = 3p$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} p \\ 3p \\ -2p \end{pmatrix} = p \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$n = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$n \cdot r = n \cdot a$$

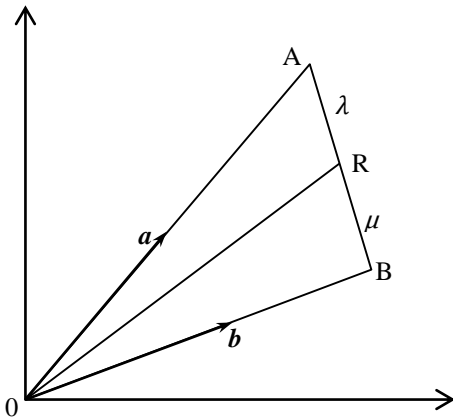
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ -11 \end{pmatrix}$$

$$x + 3y - 2z = -2 + 15 + 22$$

$$x + 3y - 2z = 35$$

### INTERNAL AND EXTERNAL DIVISIONS

Let A and B be points in space with position vectors  $\mathbf{a}$  and  $\mathbf{b}$ .



Let R be a point on a line segment AB dividing AB internally in the ratio of  $\lambda : \mu$

$$\mathbf{OR} = \mathbf{OA} + \mathbf{AR}$$

$$\mathbf{OR} = \mathbf{a} + \frac{\mu}{\lambda + \mu} \mathbf{AB}$$

$$= \mathbf{a} + \frac{\lambda}{\lambda + \mu} (\mathbf{b} - \mathbf{a})$$

$$\mathbf{OR} = \frac{a\lambda + a\mu + b\lambda - a\lambda}{\lambda + \mu}$$

$$\mathbf{OR} = \frac{a\mu + b\lambda}{\lambda + \mu}$$

#### Example I

Given that;  $OP = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$ ,  $OQ = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ . Find the coordinates

of R such that  $PR : RQ = 1:2$

$$\mathbf{r} = \frac{\mathbf{a}\mu + \mathbf{b}\lambda}{\lambda + \mu}$$

$$\mathbf{OR} = 2 \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\mathbf{OR} = \begin{pmatrix} 9 \\ -6 \\ 12 \end{pmatrix}$$

$$\mathbf{OR} = \frac{1}{3} \begin{pmatrix} 0 \\ -6 \\ 12 \end{pmatrix}$$

$$\mathbf{R} = (3, -2, 4)$$

#### Example II

The points A  $\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$  and B  $\begin{pmatrix} 7 \\ 6 \\ 1 \end{pmatrix}$  form a line segment

which is divided externally in the ratio of 4:-1. Find the coordinates of T

$$(\mathbf{OT}) = \frac{-1 \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + 4 \begin{pmatrix} 7 \\ 6 \\ 1 \end{pmatrix}}{-1 + 4}$$

$$\mathbf{OT} = \frac{\begin{pmatrix} -2 + 28 \\ 1 + 24 \\ -6 + 4 \end{pmatrix}}{3}$$

$$= \left(\frac{1}{3}\right) \begin{pmatrix} 26 \\ 25 \\ -2 \end{pmatrix}$$

$$\mathbf{OT} = \left(\frac{26}{3}, \frac{25}{3}, -\frac{2}{3}\right)$$

#### Example III

Find the position vectors  $\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$  and  $\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix}$ , Find the

position vectors of C which divides AB externally in the ratio of 5:-3

**Solution:**

$$\frac{-3 \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + 5 \begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix}}{5 + -3}$$

$$\frac{\begin{pmatrix} -9 \\ 6 \\ -15 \end{pmatrix} + \begin{pmatrix} 45 \\ 5 \\ -5 \end{pmatrix}}{2}$$

$$\frac{\begin{pmatrix} -9 + 45 \\ 6 + 5 \\ -15 - 5 \end{pmatrix}}{2}$$

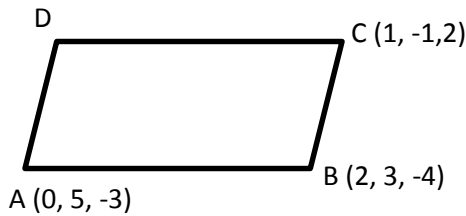
$$\frac{1}{2} \begin{pmatrix} 36 \\ 11 \\ -20 \end{pmatrix}$$

$$OC = \begin{pmatrix} 18 \\ 11/2 \\ 10 \end{pmatrix}$$

$$C = \left(18, \frac{11}{2}, -10\right)$$

#### Example IV

Given that A(0, 5, -3), B(2, 3, -4) and C(1, -1, 2). Find the coordinates of D if ABCD is a rectangle or parallelogram.



$$AB = DC$$

$$(OB - OA) = (OC - OD)$$

$$OD = OC + OA - OB$$

$$OD = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

$$OD = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

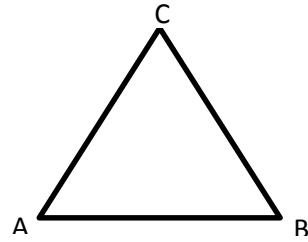
$$OD = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$D = (-1, 1, 3)$$

#### Proving that three points are vertices of a triangle

Give a triangle ABC with vertices

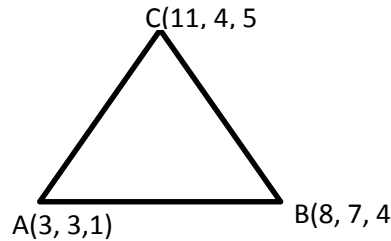
$$A = (x_1, y_1, z_1) \quad B = (x_2, y_2, z_2) \quad C = (x_3, y_3, z_3)$$



$$\begin{aligned} AB + BC + CA &= \mathbf{0} \\ OB - OA + OC - OB + OA - OC &= \mathbf{0} \end{aligned}$$

#### Example

Show that  $3i + 3j + k$ ,  $8i + 7j + 4k$  and  $11i + 4j + 5k$  are vertices of a triangle



$$AB + BC + CA = \mathbf{0}$$

$$OB - OA + OC - OB + OA - OC$$

$$= \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 11 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 11 \\ 4 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

#### Length and the equation of the perpendicular drawn from the point

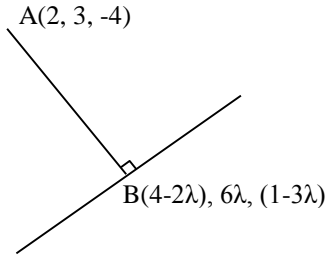
##### Example I

Find the equation and length of the perpendicular drawn from a point (2, 3, -4) to the line

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$$

##### Solution

$$\begin{aligned} \frac{4-x}{2} &= \frac{y}{6} = \frac{1-z}{3} \\ \Rightarrow \frac{x-4}{-2} &= \frac{y}{6} = \frac{z-1}{-3} \end{aligned}$$



$$r = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2\lambda \\ 6\lambda \\ -1 - 3\lambda \end{pmatrix}$$

$$AB \cdot \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix} = 0$$

$$AB = OB - OA$$

$$= \begin{pmatrix} 4 - 2\lambda - 2 \\ 6\lambda - 3 \\ 1 - 3\lambda - 4 \end{pmatrix} = \begin{pmatrix} 2 - 2\lambda \\ 6\lambda - 3 \\ 5 - 3\lambda \end{pmatrix}$$

$$\begin{pmatrix} 2 - 2\lambda \\ 6\lambda - 3 \\ 5 - 3\lambda \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix} = 0$$

$$(-2)(2 - 2\lambda) + 6(6\lambda - 3) - 3(5 - 3\lambda) = 0$$

$$-4 + 4\lambda + 36\lambda - 18 - 15 + 9\lambda = 0$$

$$36\lambda + 9\lambda + 4\lambda - 18 - 15 - 4 = 0$$

$$49\lambda = 37$$

$$\lambda = \frac{37}{49}$$

$$AB = \begin{pmatrix} 2 - 2\left(\frac{37}{49}\right) \\ 6\left(\frac{37}{49}\right) - 3 \\ 5 - 3\left(\frac{37}{49}\right) \end{pmatrix}$$

$$AB = \begin{pmatrix} 24/49 \\ 75/49 \\ 134/49 \end{pmatrix}$$

$$r = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 24/49 \\ 75/49 \\ 134/49 \end{pmatrix}$$

Equation of the perpendicular

$$\frac{x-2}{72/49} = \frac{y-3}{-69/49} = \frac{z-4}{186/49}$$

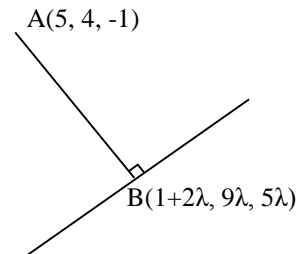
Length of the perpendicular AB

$$AB = \sqrt{\left(\frac{24}{49}\right)^2 + \left(\frac{75}{49}\right)^2 + \left(\frac{134}{49}\right)^2}$$

$$AB = 3.1719 \text{ units}$$

Find the length and equation of the perpendicular drawn from a point (5, 4, -1) to the line;  $r = i + \lambda(2i + 9j + 5k)$

**Solution**



$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2\lambda \\ 9\lambda \\ 5\lambda \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ 9\lambda \\ 5\lambda \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 + 2\lambda - 5 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix} = \begin{pmatrix} 2\lambda - 4 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix}$$

$$AB \cdot d = 0$$

$$d = \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 2\lambda - 4 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix} = 0$$

$$2(2\lambda - 4) + 9(9\lambda - 4) + 5(5\lambda + 1) = 0$$

$$4\lambda - 8 + 81\lambda - 36 + 25\lambda + 5 = 0$$

$$81\lambda + 25\lambda + 4\lambda - 8 + 5 - 36 = 0$$

$$110\lambda = 39$$

$$\lambda = \frac{39}{110}$$

$$AB = \begin{pmatrix} 2\left(\frac{39}{110}\right) - 4 \\ 9\left(\frac{39}{110}\right) - 4 \\ 5\left(\frac{39}{110}\right) + 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-362}{110} \\ \frac{-89}{110} \\ \frac{-305}{110} \end{pmatrix}$$

$$|AB| = \sqrt{\left(\frac{-362}{110}\right)^2 + \left(\frac{-89}{110}\right)^2 + \left(\frac{-305}{110}\right)^2}$$

$$|AB| = 4.379 \text{ units}$$

Equation of the perpendicular bisector is

$$r = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} \frac{-362}{110} \\ \frac{-89}{110} \\ \frac{305}{110} \end{pmatrix}$$

$$\frac{x-5}{-362/110} = \frac{y-4}{-89/110} = \frac{z+1}{305/110} = \mu$$

## Shortest Distance between Parallel Planes

### Example I

Find the perpendicular distance between two parallel planes;

$$2x + 5y - 14z = 30$$

$$2x + 5y - 14z = -15$$

**Solution**

$$r \cdot \hat{n} = d_1$$

Plane 1

$$r \cdot \left( \frac{2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}}{15} \right) = \frac{30}{15}$$

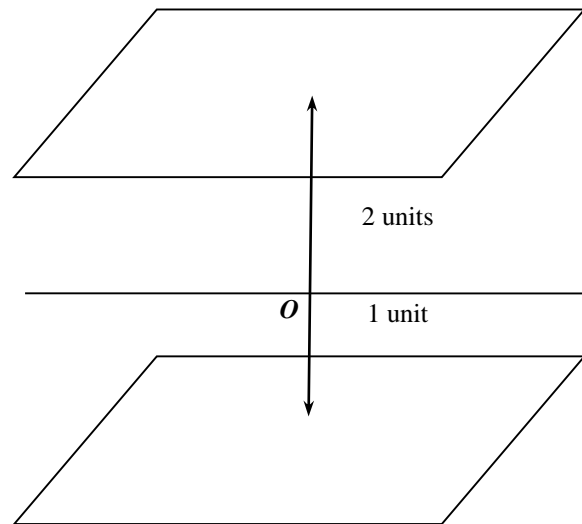
$$r \cdot \left( \frac{2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}}{15} \right) = 2$$

$$r \cdot \left( \frac{2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}}{15} \right) = 2$$

Plane 2

$$r \cdot 2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k} = -15$$

$$r \cdot \left( \frac{2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}}{15} \right) = -1$$



### Example II

Find the perpendicular distance between two parallel planes;

$$x + 2y - z = -4 \text{ and } x + 2y - z = 3$$

$$r \cdot \hat{n} = d_1$$

For plane 1

$$r \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -4$$

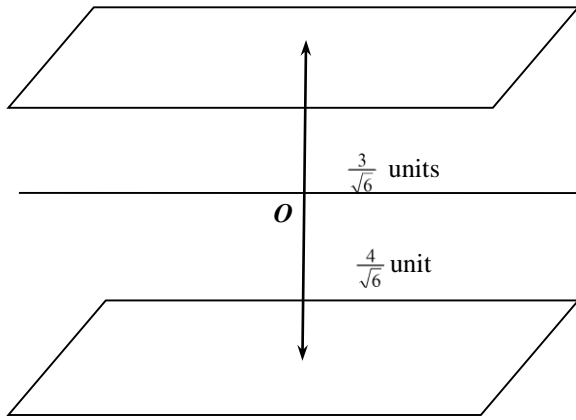
$$r \cdot \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{\sqrt{6}} = \frac{-4}{\sqrt{6}}$$

For plane 2

$$r \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 3$$

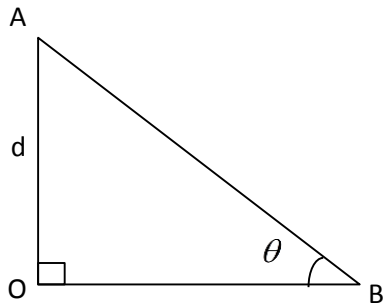
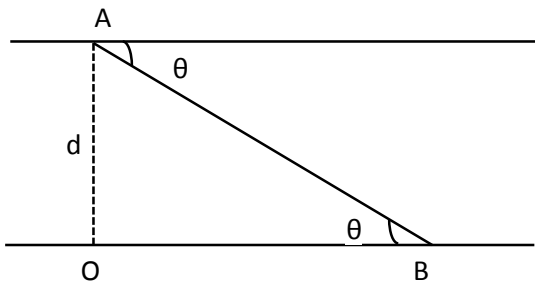
$$r \cdot \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{\sqrt{6}} = \frac{3}{\sqrt{6}}$$





$$= \frac{3}{\sqrt{6}} + \frac{4}{\sqrt{6}} = \frac{7}{\sqrt{6}} \text{ units}$$

### Shortest distance between two parallel lines



Distance between a point A and line B

$$d = AB \sin \theta$$

#### Example I

Find the shortest distance between the following pairs of parallel lines

$$\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-3}{2} \quad \text{and}$$

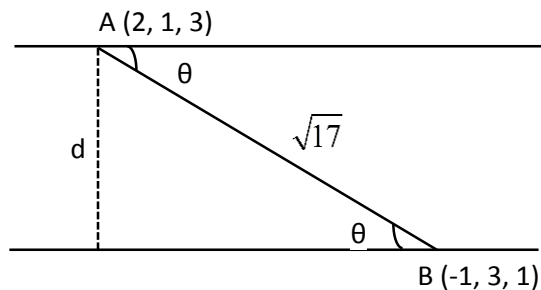
$$\frac{x+1}{1} = \frac{y-3}{-1} = \frac{z-1}{2}$$

$$AB = \sqrt{(2+1)^2 + (1-3)^2 + (3-1)^2}$$

$$AB = \sqrt{17}$$

$$\overline{AB} = \overline{OB} - \overline{OA}$$

$$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix}$$



$$\cos \theta = \frac{AB \cdot d}{|AB| \cdot d}$$

$$\cos \theta = \frac{\begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}{\sqrt{17} \sqrt{6}}$$

$$\theta = \cos^{-1} \left( \frac{-9}{\sqrt{100}} \right)$$

$$\theta = 26.8^\circ$$

$$\sin 26.8^\circ = \frac{d}{\sqrt{17}}$$

$$d = 1.859 \text{ units}$$

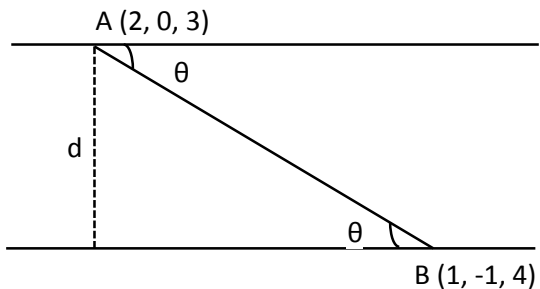
#### Example II

Find the distance between the following pairs of parallel lines

$$r = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$r = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

**Solution**



$$AB = \sqrt{(2-1)^2 + (0+1)^2 + (3-4)^2}$$

$$AB = \sqrt{1+1+1}$$

$$AB = \sqrt{3}$$

$$\cos \theta = \frac{2}{\sqrt{18}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{18}}\right)$$

$$\theta = 61.9^\circ$$

$$\sin \theta = \frac{d}{\sqrt{3}}$$

$$\sin 61.9^\circ = \frac{d}{\sqrt{3}}$$

$$d = \sqrt{3} \sin 61.9^\circ$$

$$d = 1.52789 \text{ units}$$

### SKEW LINES

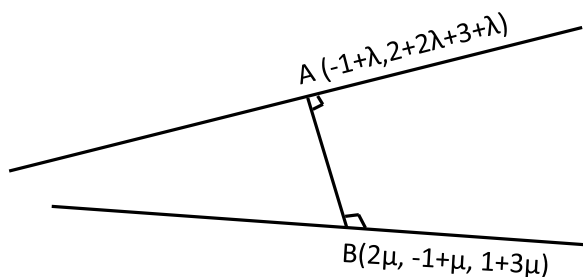
These are lines which are neither parallel nor perpendicular

Shortest distance between two skew lines

#### Example I

Find the shortest distance between the following skew lines

$$\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$



$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$\begin{pmatrix} 2\mu - (1 + \lambda) \\ -1 + \mu - (2 + 2\lambda) \\ 1 + 3\mu - (3 + \lambda) \end{pmatrix} \cdot \begin{pmatrix} 2\mu - \lambda + 1 \\ \mu - 2\lambda - 3 \\ 3\mu - \lambda - 2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2\mu - \lambda + 1 \\ \mu - 2\lambda - 3 \\ 3\mu - \lambda - 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$2\mu - \lambda + 1 + 2\mu - 4\lambda - 6 + 3\mu - \lambda - 2 = 0$$

$$7\mu - 6\lambda = 7 \dots \dots \dots (1)$$

$$\begin{pmatrix} 2\mu - \lambda + 1 \\ \mu - 2\lambda - 3 \\ 3\mu - \lambda - 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$4\mu - 2\lambda + 2 + \mu - 2\lambda - 3 + 9\mu - 3\lambda - 6 = 0$$

$$14\mu - 7\lambda - 7 = 0$$

$$14\mu - 7\lambda = 7 \dots \dots \dots (2)$$

$$\mu = \frac{-1}{5}, \lambda = -\frac{7}{5}$$

$$AB = \begin{pmatrix} 2 \\ -0.4 \\ -1.2 \end{pmatrix}$$

$$AB = \sqrt{2^2 + (-0.4)^2 + (-1.2)^2} = 2.3664 \text{ units}$$

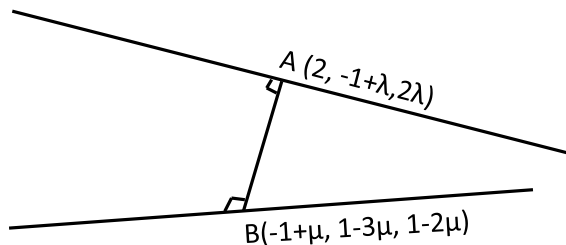
#### Example II

Find the shortest distance between the following pairs of skew lines

$$\frac{x-2}{0} = \frac{y+1}{1} = \frac{z}{2} \text{ and } \frac{x+1}{1} = \frac{y-1}{-3} = \frac{z-1}{-2}$$

#### Solution

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} (-1+\mu)-2 \\ (1-3\mu)-(-1+\lambda) \\ (1-2\mu)-2\lambda \end{pmatrix} = \begin{pmatrix} \mu-3 \\ -3\mu-\lambda+2 \\ 1-2\mu-2\lambda \end{pmatrix}$$

$$\begin{pmatrix} \mu-3 \\ -3\mu-\lambda+2 \\ 1-2\mu-2\lambda \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$-3\mu - \lambda + 2 + 2 - 4\mu - 4\lambda = 0$$

$$-7\mu - 5\lambda - 4 = 0$$

$$7\mu + 5\lambda - 4 = 0$$

$$7\mu + 5\lambda = 4 \dots \dots \dots (1)$$

$$\begin{pmatrix} \mu-3 \\ -3\mu-\lambda+2 \\ 1-2\mu-2\lambda \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = 0$$

$$\mu - 3 + 9\mu + 3\lambda - 6 - 2 + 4\mu + 4\lambda$$

$$14\mu + 7\lambda - 11 = 0$$

$$14\mu + 7\lambda = 11 \dots \dots \dots (2)$$

$$\mu = \frac{9}{7}, \lambda = -1$$

$$AB = \sqrt{\left(\frac{-12}{7}\right)^2 + \left(\frac{-6}{7}\right)^2 + \left(\frac{3}{7}\right)^2}$$

$$AB = \sqrt{\frac{144}{49} + \frac{36}{49} + \frac{9}{49}}$$

$$AB = \frac{3\sqrt{21}}{7} \text{ units}$$

### Vector Geometry

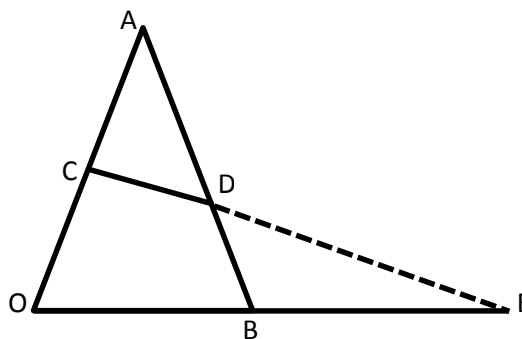
#### Example I

Triangle OAB has  $OA = \mathbf{a}$ ,  $OB = \mathbf{b}$ . C is a point on OA such that  $OC = \frac{2}{3}\mathbf{a}$ . D is a mid point of AB when CD is

produced, it meets OB at E such that  $DE = nCD$  and  $BE = k\mathbf{b}$ . Express BE, DE in terms of;

a)  $\mathbf{n}$ ,  $\mathbf{a}$  and  $\mathbf{b}$

b)  $\mathbf{k}$ ,  $\mathbf{b}$  and  $\mathbf{a}$ . Hence find the values of  $\mathbf{n}$  and  $\mathbf{k}$ .



$$\overrightarrow{DE} = n\overrightarrow{CD}$$

$$\overrightarrow{DE} = n[\overrightarrow{CA} + \overrightarrow{AD}]$$

$$\overrightarrow{DE} = n\left[\frac{1}{3}\mathbf{a} + \overrightarrow{AD}\right]$$

$$\overrightarrow{DE} = n\left[\frac{1}{3}\mathbf{a} + \frac{1}{2}\overrightarrow{AB}\right]$$

$$\overrightarrow{DE} = \frac{1}{3}n\mathbf{a} + \frac{1}{2}n\mathbf{b} - \frac{1}{2}n\mathbf{a}$$

$$\overrightarrow{DE} = \frac{-1}{6}n\mathbf{a} + \frac{1}{2}n\mathbf{b} \dots \dots \dots (1)$$

$$\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE}$$

$$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB} + k\mathbf{b}$$

$$\overrightarrow{DE} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) + k\mathbf{b}$$

$$\overrightarrow{DE} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} + k\mathbf{b}$$

$$\overrightarrow{DE} = \left(\frac{1}{2} + k\right)\mathbf{b} - \frac{1}{2}\mathbf{a}$$

$$-\frac{1}{2}\mathbf{a} = -\frac{1}{6}n\mathbf{a}$$

$$\frac{1}{2} = \frac{1}{6}n$$

$$6 = 2n$$

$$n = 3$$

$$\left(\frac{1}{2} + k\right)\mathbf{b} = \frac{1}{2}n\mathbf{b}$$

$$\frac{1}{2} + k = \frac{1}{2} \times 3$$

$$k = \frac{3}{2} - \frac{1}{2} = 1$$

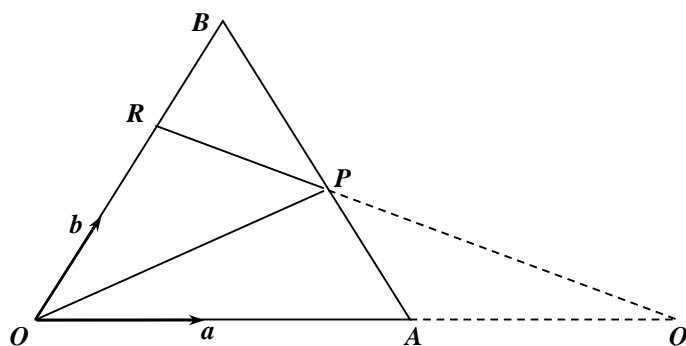
### Example II

Given that OA is  $\mathbf{a}$  and OB= $\mathbf{b}$  point R is on OB such that OR:RB=4:1. Point P is on AB such that BP:PA=2:3.

When RP and OA are both produced, they meet at Q. Find OR and OP in terms of  $\mathbf{a}$  and  $\mathbf{b}$

ii) OQ in terms of  $\mathbf{a}$

**Solution**



$$\overline{OR} = \frac{4}{5}OB \quad \Rightarrow \quad \overline{OR} = \frac{4}{5}\mathbf{b}$$

$$\overline{OP} = \overline{OB} + \overline{BP}$$

$$\overline{OP} = \mathbf{b} + \frac{2}{5}\overline{BA}$$

$$\overline{OP} = \mathbf{b} + \frac{2}{5}(\mathbf{a} - \mathbf{b})$$

$$\overline{OP} = \frac{1}{5}(3\mathbf{b} + 2\mathbf{a})$$

$$\overline{OQ} = \lambda\overline{OA} = \lambda\mathbf{a}$$

$$\overline{OQ} = \overline{OR} + \overline{RQ}$$

$$\overline{OQ} = \frac{4}{5}\mathbf{b} + \mu\overline{RP}$$

$$\overline{OQ} = \frac{4}{5}\mathbf{b} + \mu\left(\frac{-4}{5}\mathbf{b} + \frac{1}{5}(2\mathbf{a} + 3\mathbf{b})\right)$$

$$\overline{OQ} = \left(\frac{4}{5} - \frac{1}{5}\mu\right)\mathbf{b} + \frac{2}{5}\mu\mathbf{a}$$

$$\frac{4}{5} - \frac{1}{5}\mu = 0$$

$$\mu = 4$$

$$\lambda = \frac{2}{5}\mu$$

$$\lambda = \frac{8}{5}$$

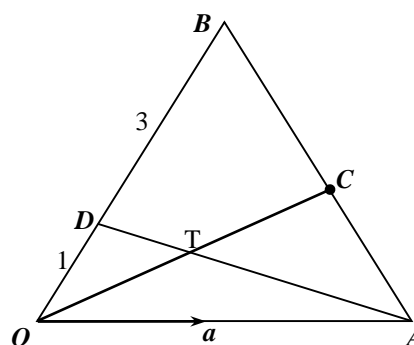
$$OQ = \frac{8}{5}\mathbf{a}$$

### Example III

O, A and B are non collinear points OA =  $\mathbf{a}$ , OB =  $\mathbf{b}$ , C is midpoint of AB, D is a point on OB such that  $\overline{OD} = \frac{1}{4}\overline{OB}$ .

T is a point of intersection of OC and AD. Find the vector OT in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

**Solution**



$$\overline{OT} = \lambda\overline{OC}$$

$$\overline{OC} = \overline{OB} + \overline{BC}$$

$$= \mathbf{b} + \frac{1}{2}\overline{BA}$$

$$= \mathbf{b} + \frac{1}{2}(\mathbf{a} - \mathbf{b})$$

$$\overline{OC} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$\overline{OT} = \lambda\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$$

$$\overline{OT} = \frac{1}{2}\lambda\mathbf{a} + \frac{1}{2}\lambda\mathbf{b} \dots \dots \dots (i)$$

$$\overline{OT} = \overline{OA} + \overline{AT}$$

$$= \mathbf{a} + \mu\overline{AD}$$

$$\overline{AD} = \overline{AO} + \overline{OD}$$

$$= \mathbf{a} + \frac{1}{4}\mathbf{b}$$

$$\overline{OT} = \mathbf{a} + \mu\left(\mathbf{a} + \frac{1}{4}\mathbf{b}\right)$$

$$\overline{OT} = \mathbf{a} - \mu\mathbf{a} + \frac{1}{4}\mu\mathbf{b}$$

$$\mathbf{OT} = (1 - \mu)\mathbf{a} + \frac{1}{4}\mu\mathbf{b} \dots\dots\dots \text{(ii)}$$

Equating components of vectors  $\mathbf{a}$  and  $\mathbf{b}$  in Eqns (i) and (ii);

$$\frac{1}{2}\lambda = 1 - \mu \dots\dots\dots \text{(iii)}$$

$$\frac{1}{2}\lambda = \frac{1}{4}\mu \dots\dots\dots \text{(iv)}$$

From Eqn (iv);

$$2\lambda = \mu$$

$$\Rightarrow \frac{\lambda}{2} = 1 - 2\mu$$

$$\frac{5\lambda}{2} = 1$$

$$\lambda = \frac{2}{5}$$

$$\mu = \frac{4}{5}$$

$$\mathbf{OT} = \frac{2}{5}\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$$

$$\mathbf{OT} = \frac{1}{5}(\mathbf{a} + \mathbf{b})$$

## Revision Exercise

1. In a triangle  $ABC$ , the altitudes from  $B$  and  $C$  meet the opposite sides at  $E$  and  $F$  respectively.  $BE$  and  $CF$  intersect at  $O$ . Taking  $O$  as the origin, use the dot product to prove that  $AO$  is perpendicular to  $BC$

(b) Find the point of intersection of the line

$$\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4} \text{ with the plane}$$

$$3x + 4y + 2z - 25 = 0$$

(c) Find the angle between the line  $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$

$$\text{and the plane } 4x + 3y + 1 = 0$$

2. (a) Show that the equation of the plane through points  $A$  with position vector  $2\mathbf{i} + 2\mathbf{k}$  perpendicular to the vector  $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$  is  $x + 3y - 2z + 10 = 0$

(b) (i) Show that the vector  $2\mathbf{i} - 5\mathbf{j} + 3.5\mathbf{k}$  is

perpendicular to the line  $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \lambda(4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$

(ii) Calculate the angle between the vector  $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$  and the line in (b)(i) above.

3. A point  $P$  has coordinates  $(1, -2, 3)$  and a certain plane has the equation  $x + 2y + 2z = 8$ . The line

through  $P$  parallel to the line  $\frac{x}{3} = \frac{y+1}{-1} = \frac{z+1}{-2}$

meets the plane at a point  $Q$ .

4. (a) The line through  $A(1, -2, 2)$  and perpendicular to the plane  $4x - y + 2z + 12 = 0$  meets the plane in point  $B$ . Find the coordinates of  $B$ .

(b) Given that the vectors  $a\mathbf{i} - 2\mathbf{j} + \mathbf{k}$  and  $2a\mathbf{i} + a\mathbf{j} - 4\mathbf{i}$  are perpendicular, find the values of  $a$ .

5. Find the equation of the plane through the point  $(1, 2, 3)$  and perpendicular to the vector  $\mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ .

6. (a) The vertices of a triangle are  $P(2, -1, 5)$ ,  $Q(7, 1, -3)$  and  $R(13, -2, 0)$ . Show that  $\angle PQR = 90^\circ$ . Find the coordinates of  $S$  if  $PQRS$  is a rectangle.

(b) Find the equation of the line through  $A(2, 2, 5)$  and  $B(1, 2, 3)$

(c) If the line in (b) above meets the line

$$\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3} \text{ at } P, \text{ find the:}$$

(i) coordinates of  $P$ ,

(ii) angle between the two lines

7. The position vector of points  $P$  and  $Q$  are  $2\mathbf{i} - 3\mathbf{j}$  and  $3\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}$  respectively. Determine the length of  $PQ$ .  $PQ$  meets the plane  $4x + 5y - 2z = 5$  at point  $S$ . Find:

(a) the coordinates of  $S$ ,

(b) the angle between  $PQ$  and the plane.

8. (a) Find the angle between the line  $\mathbf{r} = 3\mathbf{k} + \lambda(7\mathbf{i} - \mathbf{j} + 4\mathbf{k})$  and the plane  $\mathbf{r} \cdot (2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}) = 8$

(b) Show that the lines with vector equations

$$\mathbf{r}_1 = (1 + 4\lambda)\mathbf{i} + (1 - \lambda)\mathbf{j} + (2\lambda)\mathbf{k}, \text{ and}$$

$$\mathbf{r}_2 = (5 + 3\mu)\mathbf{i} + (2\mu)\mathbf{j} + (2 - 5\mu)\mathbf{k}.$$

intersect at right angles and give the position vector of the point of intersection.

9. Find the equation of the line with directrix vector  $\mathbf{d}$  which passes through the point with position vector  $\mathbf{a}$  given that

$$\text{(a) } \mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{d} = 3\mathbf{i} - \mathbf{k}$$

$$\text{(b) } \mathbf{a} = 4\mathbf{i} - 3\mathbf{k}, \quad \mathbf{d} = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

10. Find the vector equation of the line which passes through the points with (a) position vectors  $3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$  and  $-2\mathbf{j} + \mathbf{j} + \mathbf{k}$ .

(a) position vector  $\mathbf{i} + 4\mathbf{j}$  and  $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ ,

(b) coordinates  $(0, 6, -6)$  and  $(5, -7, 2)$

(c) coordinates  $(0, 0, 0)$  and  $(5, -2, 3)$

11. Write down in parametric form the vector equations of the planes through the given points parallel to the given pairs of vectors.

(a)  $(1, -2, 0)$ ;  $\mathbf{i} + 3\mathbf{j}$  and  $-\mathbf{j} + 2\mathbf{k}$

(b) the origin;  $2\mathbf{i} - \mathbf{j}$  and  $-\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$

(c)  $(3, 1, -1)$ ;  $\mathbf{j}$  and  $\mathbf{i} + \mathbf{j} + \mathbf{k}$ .

12. Find a vector equation for the plane passing through the points with position vectors  $2k$ ,  $i - 3j + k$  and  $5i + 2j$ .
13. Find the vector equation of the plane through the points  $A(1, 0, -2)$  and  $B(3, -1, 1)$  which is parallel to the line with vector equation  $\mathbf{r} = 3i + (2\lambda - 1)j + (5 - \lambda)k$ . Hence find the coordinates of the point of intersection of the plane and the line  $\mathbf{r} = \mu i + (5 - \mu)j + 2\mu - 7k$ .
14. Find a vector equation for the line joining the points  
 (a)  $(2, 6)$  and  $(5, 2)$   
 (b)  $(-1, 2, -3)$  and  $(6, 3, 0)$ .
15. (a) Points  $A$  and  $B$  have coordinates  $(4, 1)$  and  $(2, -5)$  respectively. Find a vector equation for the line which passes through  $A$  and perpendicular to the line  $AB$ .  
 (b) Points  $P$  and  $Q$  have coordinates  $(3, 5)$  and  $(-3, -7)$  respectively. Find a vector equation for the line which passes through the point  $P$  and which is perpendicular to the line  $PQ$ .
16. Find a vector equation for the perpendicular bisector of the points:  
 (a)  $(6, 3)$  and  $(2, -5)$   
 (b)  $(7, -1)$  and  $(3, -3)$ .
17. Points  $P$ ,  $Q$  and  $R$  have position vectors  $4i - 4j$ ,  $2i + 2j$ , and  $8i + 6j$  respectively.  
 (a) Find a vector equation for the line  $L_1$  which is the perpendicular bisector to the points  $P$  and  $Q$   
 (b) Find a vector equation for the line  $L_2$  which is the perpendicular bisector to the points  $A$  and  $R$ .  
 (c) Hence find the position vector of the point where  $L_1$  and  $L_2$  meet.
18. Two lines  $L_1$  and  $L_2$  have equations  

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \text{ and } L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$
 (a) Show that  $L_1$  and  $L_2$  are concurrent (meet at a common point) and find the position vector of their point of intersection.  
 (b) Find the angle between  $L_1$  and  $L_2$ .
19. Points  $P$ ,  $Q$ , and  $R$  have coordinates  $(-1, 1)$ ,  $(4, 6)$  and  $(7, 3)$  respectively.  
 (a) Show that the perpendicular distance from the point  $R$  to the line  $PQ$  is  $3\sqrt{2}$ .  
 (b) Deduce that the area of the triangle  $PQR$  is 15 sq.units.
20. Points  $A$ ,  $B$  and  $C$  have position vectors  $-i + 3j + 9k$ ,  $5i + 6j - 4k$  and  $4i + 7j + 5k$  respectively.  $P$  is the point on  $AB$  such that  $\overrightarrow{AP} = \lambda \overrightarrow{AB}$ . Find:  
 (a)  $\overrightarrow{AB}$   
 (b)  $\overrightarrow{CP}$   
 (c) Find the perpendicular distance from the point  $C$  to the line  $AB$ .
21. Two lines  $L_1$  and  $L_2$  have vector equations  
 $\mathbf{r}_1 = (2 - 3\lambda)\mathbf{i} + (1 + \lambda)\mathbf{j} + 4\lambda\mathbf{k}$   
 $\mathbf{r}_2 = (-1 + 3\lambda)\mathbf{i} + 3\mathbf{j} + (4 - \lambda)\mathbf{k}$  respectively. Find:  
 (a) the position vector of their common point of intersection.  
 (b) the angle between the lines.
22. Find the equation of the plane containing points  $P(1, 1, 1)$ ,  $Q(1, 2, 0)$  and  $(-1, 2, 1)$ .
23. Find the equation of the plane containing point  $(4, -2, 3)$  and parallel to the plane  $3x - 7z = 12$
24. Show that the point with position vector  $7i - 5j - 4k$  lies in the plane  $\mathbf{r} = 4i + 3j + 2k + \lambda(i - j - k) + \mu(2i + 3j + k)$ . Find the point at which the line  $x = y - 1 = 2z$  intersects the plane  $4x - y + 3z = 8$ .
25. Find the parametric equations for the line through the point  $(0, 1, 2)$  that is parallel to the plane  $x + y + z = 2$  and perpendicular to the line  $x = 1 + t$ ,  $y = 1 - t$ ,  $z = 2t$ .
26. Find the distance between the parallel planes  
 $z = x + 2y + 1$  and  $3x + 6y - 3z = 4$
27. Two planes are given by the parametric equations  
 $x = r + 3$  and  $x = 1 + r + s$   
 $y = 3s$  and  $y = 2 + r$   
 $z = 2r$  and  $z = -3 + 5$   
 Find the Cartesian equation of the intersection point.
28. The equation of a plane  $P$  is given by  $r \cdot \begin{pmatrix} 2 \\ 6 \\ 9 \end{pmatrix} = 33$ , where  $r$  is the position vector of  $P$ . find the perpendicular distance from the plane to the origin.
29. The line through point  $P(1, -2, 3)$  and parallel to the line  $\frac{x}{3} + \frac{y+1}{-1} = z + 1$  meets the plane  $x + 2y + 27z = 8$  at  $Q$ . find the coordinates of  $Q$ .
30. (a) Find the angle between the plane  $x + 4y - z = 72$  and the line  $\mathbf{r} = 9i + 6j + 8k$ .

- (b) obtain the equation of the plane that passes through  $(1, -2, 2)$  and perpendicular to the line  $\frac{x-9}{4} = \frac{y-6}{-1} = \frac{z-8}{1}$
- (c) Find the parametric equations of the line of intersection of the plane  $x + y + z = 4$  and  $x - y + 2z + 2 = 0$
31. Find the point of intersection of the three planes  $2x - y + 3z = 4$ ,  $3x - 2y + 6z = 3$  and  $7x - 4y + 5z = 11$ .
32. Find the Cartesian equation of the plane with parametric vector equation  $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
33. Find the Cartesian equation of the plane containing the point with position vector  $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$  and parallel to the vectors  $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ .
34. Find the Cartesian equation of the plane containing the points with position vectors  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ .
35. Find the perpendicular distance from the plane  $\mathbf{r} \cdot (2\mathbf{i} - 14\mathbf{j} + 5\mathbf{k}) = 10$  to the origin.
36. Find the position vector of the point where the line  $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$  meets the plane  $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = 15$ .
37. Two lines have vector equations  $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$  and  $\mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ . Find the position vector of the point of intersection of the two lines and the Cartesian equation of the plane containing the two lines.
38. The position vector of points P and Q are  $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$  and  $2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ , respectively. Find the acute angle between PQ and the line  $1 - x = \frac{y-3}{2} = \frac{4-z}{4}$ .
- (b) Find the point of intersection of the line  $x - 2 = 2y + 1 = 3 - z$  and the plane  $x + 2y + z = 3$ .
- (c) Find the equation of the plane through the origin parallel to the lines  $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k} + s(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$  and  $\mathbf{r} = 4\mathbf{i} - 5\mathbf{j} - 8\mathbf{k} + t(3\mathbf{i} + 7\mathbf{j} - 6\mathbf{k})$
39. (a) The points A and B have position vectors  $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$  and  $\mathbf{b} = 7\mathbf{i} - 6\mathbf{j} + \mathbf{k}$  respectively. Find the coordinates of a point P which divides the vector AB in the ratio:
- (i) 4:1  
(ii) 1:4
40. (b) Find the Cartesian equation of the plane through the origin parallel to the lines  $x - 3 = 3 - y = \frac{z+1}{-2}$  and  $\frac{x-4}{3} = \frac{y+5}{7} = \frac{x+8}{-6}$
- (c) Find the angle between the line  $1 - x = \frac{y-3}{2} = \frac{4-z}{4}$  and the plane  $2x - 3y - 2z + 5 = 0$ .
41. (a) Determine the unit vector perpendicular to the plane containing the points A(0, 2, -4), B(2, 0, 2) and C(-8, 4, 0).
- (b) Find the equation of the plane in (a) above
- (c) Show that the point T(5, -4, 3) lies on the plane in (a) above.
- (d) Write down the equation in the form  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$  of the perpendicular through the point P(3, 4, 2) to the plane in (a) above.
- (e) If the perpendicular meets the plane in (a) above at N, determine vector NP.